Pressure Gradient Effects on Laminar Boundary Layers at Large Prandtl Numbers

R. Gopinath* and A. V. Murthy*
National Aeronautical Laboratory, Bangalore, India

Abstract

THE well-known method of matched asymptotic expansions is used to study the effect of pressure gradient on compressible, laminar boundary layers at large Prandtl numbers. For compressible flow, the momentum and energy equations are, in general, coupled. In order to uncouple them so as to make the anlaysis tractable, a "Falkner-Skan" type of flow is considered and the effect of pressure gradient is studied for both wall-insulated and heat-transfer cases. An analysis of the energy equation near the wall, using a straightforward expansion for the enthalpy in terms of a small parameter, defined as the reciprocal of the Prandtl number, revealed a logarithmic behavior for the enthalpy, calling for a logarithmic term in the inner expansion. For the case of zero pressure gradient, a similar analysis has been made by Narasimha and Vasantha.[1] Estimates of enthalpy gradient at the wall and recovery enthalpy are made using the analytical expressions derived in this study and compared with numerical solutions for a typical value of the pressure gradient parameter. The velocity and enthalpy profiles are computed for zero as well as some typical value of the pressure gradient parameter for wall-insulated and heat-transfer cases and then compared.

Contents

The problem considered is that of studying pressure gradient effects on compressible, laminar boundary layers at large Prandtl numbers. In order to simplify the analysis and make it tractable, the following assumptions are made: 1) a linear viscosity-temperature relation; 2) local similarity; 3) Falkner-Skan type of flow for which similarity solutions exist, and with dissipation; and 4) Prandtl and Mach number parameters are constants. Using the Dorodnitsyn transformation, the basic governing equation then reduces to the following form:

$$f''' + ff'' + \beta(1-f'^2) = 0 \qquad (1)$$

$$g'' + \sigma f g' + C(\sigma - 1)(f'f'')' = 0 \qquad (2)$$

where β is the pressure gradient parameter, g the enthalpy ratio (H/H_e), σ the Prandtl number, and C a Mach number parameter (u_e^2/He). For large Prandtl numbers, a small parameter ϵ, defined as the reciprocal of the Prandtl number, is introduced and Eq. (2) may be rewritten as

$$\epsilon g'' + fg' + C(1-\epsilon)(f'f'')' = 0 \qquad (3)$$

This equation is solved by the well-known inner-outer expansion technique. The results of the analysis for recovery enthalpy, enthalpy gradient at the wall, and recovery factor r are:

$$g_r \sim 1 + Ch(0) + \frac{16Ca_3}{9}\ln\frac{a_2\sigma}{3!} + 12C\left(\frac{a_2}{6}\right)^{4/3} gi(1/3,\infty)\sigma^{1/3} + \ldots \qquad (4)$$

$$g_w' \sim \frac{(36a_2)^{1/3}}{2\Gamma(1/3)}\left\{g_r - g_w + \frac{B_1 a_3}{2a_2}\,\frac{\Gamma(5/3)}{(6a_2^2)^{1/3}}\right\} + \ldots \qquad (5)$$

$$r \sim 1 + 2h(0) + \frac{32a_3}{9}\ln\frac{a_2\sigma}{3!} + 24\left(\frac{a_2}{6}\right)^{4/3} gi(1/3,\infty)\sigma^{1/3} + \ldots \qquad (6)$$

where

g_r = recovery enthalpy

$$h(0) = \int_0^\infty\left\{\frac{(f'f'')'}{f} - \frac{2a_2}{\eta^2} - \frac{16a_3}{3\eta(\eta+1)}\right\}d\eta$$

$a_3 = -\beta$

a_2 = wall shear (known for Falkner-Skan type of flow)

g_w' = enthalpy gradient at the wall

$$B_1 = \frac{(36a_2)^{1/3}}{2\Gamma(1/3)}\left\{12C(a_2/6)^{4/3} gi(1/3,\infty) - \Theta_w\right\}$$

$\Theta = = g\epsilon^{1/3}$

The values of the recovery enthalpy and enthalpy gradient at the wall have been estimated using Eqs. (4) and (5) and compared with exact numerical solutions of the governing equation (2), which is integrated using a fourth-order Runge-Kutta technique with g_w' and g_r determined using the "Newton-Raphson" iteration technique. The computations are carried out for Prandtl numbers up to 30 and $C=(5/6)$. These results are shown in Table 1.

Generally, the estimates are higher compared to numerical solutions, but this difference progressively decreases with an increase in Prandtl number; and, at $\sigma = 30$, the difference is about 10%. It may be noted that the solution for recovery enthalpy and enthalpy gradient at the wall reduce to those of Ref. 1 when the pressure gradient is zero. A comparison of the estimates at $\beta = 0.2$ with flat plate values shows that pressure gradient increases g_w' and g_r by 45% (≃) and 30% (≃), respectively, at $\sigma = 30$, for instance.

A few words on comparison with other methods would be in order at this stage. The series method due to Meksyn, Evans, etc., form the other class of methods for solving the boundary-layer equations. Meksyn's asymptotic method usually works well for incompressible, similar flows.[2] For compressible flow, the momentum and energy equations are, in general, coupled. Even for the relatively simple case for which similar solutions exist, Meksyn's procedure leads to the requirement that two series need to be satisfied simultaneously. Considerable ingenuity it seems is often needed to improve the

Received Aug. 7, 1978; synoptic received May 7, 1979. Full paper available from National Technical Information Service, Springfield, Va. 22151, as N79-25345 at the standard price (available upon request). Copyright © American Institute of Aeronautics and Astronautics, Inc., 1979. All rights reserved.

Index category: Boundary Layers and Convective Heat Transfer —Laminar.

*Scientist, Aerodynamics Division, WTC, Belur.

Table 1 Comparison of estimates of g_w' and g_r with exact numerical solutions and flat plate values

	g_w'			g_r		
	$\beta=0.2$		$\beta=0$	$\beta=0.2$		$\beta=0$
σ	Estimate Eq. (5)	Exact (num.)	Ref. 1	Estimate Eq. (4)	Exact (num.)	Ref. 1
1	0.670	0.406	0.383	1.498	1.000	1.000
10	2.620	2.241	1.809	2.522	2.165	1.954
15	3.500	3.055	2.647	2.891	2.531	2.241
30	5.707	5.186	3.931	3.651	3.324	2.842

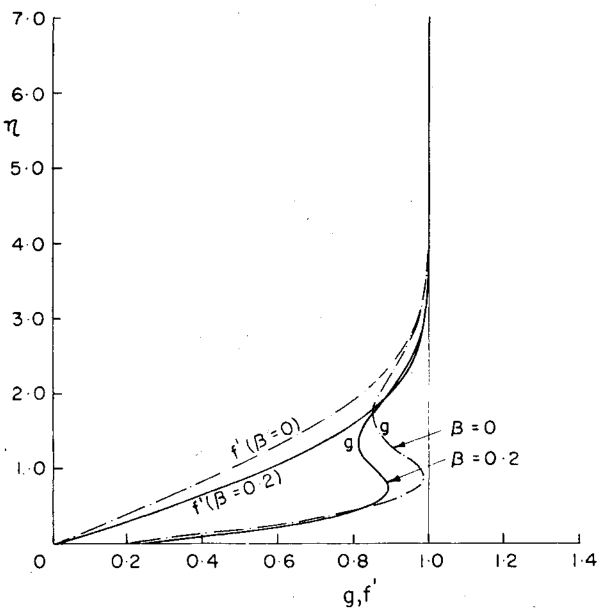

Fig. 1 Effect of pressure gradient on velocity and enthalpy profiles at $\sigma=10$ for the case of heat transfer, $g_w = 0.2$.

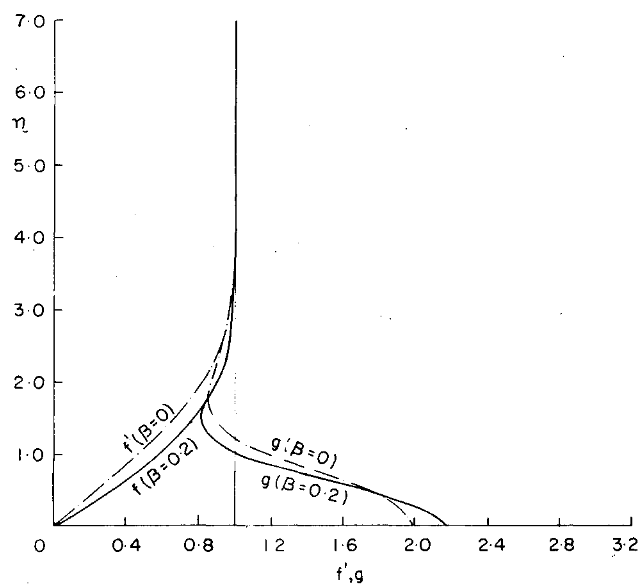

Fig. 2 Effect of pressure gradient on velocity and enthalpy profiles at $\sigma=10$ for the wall-insulated case.

approximations. So far, this method has been used only to solve the energy equation for flow on a flat plate with $\sigma \approx 1$.

Evans' method has been applied only for incompressible flows with no dissipation at large Prandtl numbers. With dissipation, for a compressible flow, there may be difficulties in the evaluation of coefficients in the series and problems of convergence.

Typical enthalpy and velocity profiles obtained by numerical computation of Eqs. (1) and (2) have been presented in Figs. 1 and 2 for a Prandtl number of 10. The interesting features of these plots are the "hill-hollow" behavior of the enthalpy for the heat-transfer case (Fig. 1) and an undershoot for the wall-insulated case (Fig. 2) for $\sigma > 1$.

This phenomenon is due primarily to large Prandtl numbers when shown on an enthalpy plot and not pressure gradient as a similar behavior may be seen even for the flat plate.

Now, a large Prandtl number implies negligible diffusion of heat over the majority of the boundary layer and, since dissipation is greatest near the wall, it follows that the temperature of the fluid increases rapidly on approaching the wall and increases with Prandtl number if the wall is insulated. The actual rise in temperature depends on the balance between the rate at which heat is being supplied to the layer by dissipation and the rate at which heat is being transported by conduction and convection. Away from the wall, the temperature of the freestream is reached over a thickness that is inversely proportional to the square root of the Prandtl number, but as the velocity layer is much thicker than the thermal layer, the velocity would not have reached the freestream value. The sum total of the kinetic energy and internal energy thus remains less than its value at the edge of the boundary layer and hence an "undershoot" in the enthalpy profile. This undershoot becomes accentuated with an increase in Prandtl number. It is interesting to note that this is almost a mirror image of the enthalpy profile for $\sigma < 1$ (Ref. 3) and can be explained using similar arguments.

In the case of heat transfer, the maximum temperature occurs near but not at the wall as the latter is kept at some temperature less than the recovery temperature, again due to dissipation. Away from the wall, convective forces begin to dominate and there is an intermediate region where the enthalpy ratio decreases to a minimum before it starts increasing to reach the asymptotic value. This explains the "hill-hollow" behavior of enthalpy for $\sigma \gg 1$. However, the same enthalpy profiles when reduced to temperature plots do not exhibit this behavior. On the other hand, they show the familiar overshoot in temperature followed by a gradual decrease to the asymptotic value. The peaks in the enthalpy profile happen to be the characteristic features at large Prandtl numbers and move closer toward the surface with an increase in Prandtl number.

References

[1] Narasimha, R. and Vasantha, S. S., "Laminar Boundary Layer on Flat Plate at High Prandtl Number," *ZAMP*, Vol. 17, 1966, pp. 585-592.

[2] Chao, B.T., *Advanced Heat Transfer*, University of Illinois Press, Urbana, Ill., 1969, pp. 54-72.

[3] Rogers, D.F., "Adiabatic Wall Solutions for the Hypersonic Compressible Laminar Boundary Layer Equations," *Physics of Fluids*, Vol. 13, No. 5, 1970, pp. 1142-1145.